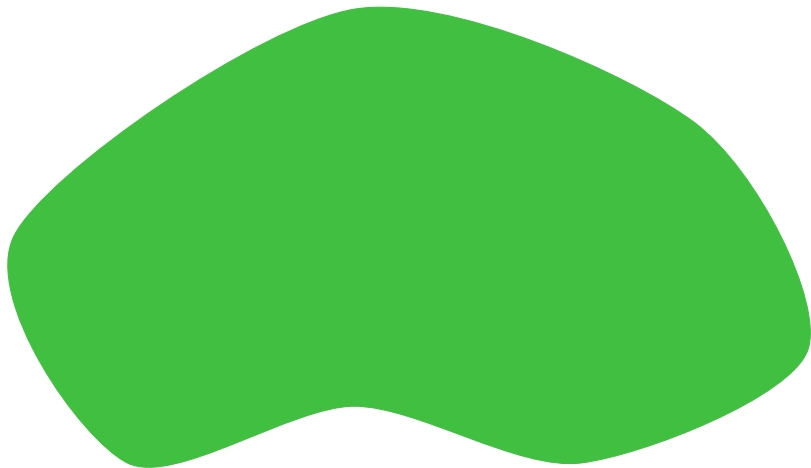


# $b$ -stable set interdiction on bipartite graphs

Stephen Chestnut & Rico Zenklusen

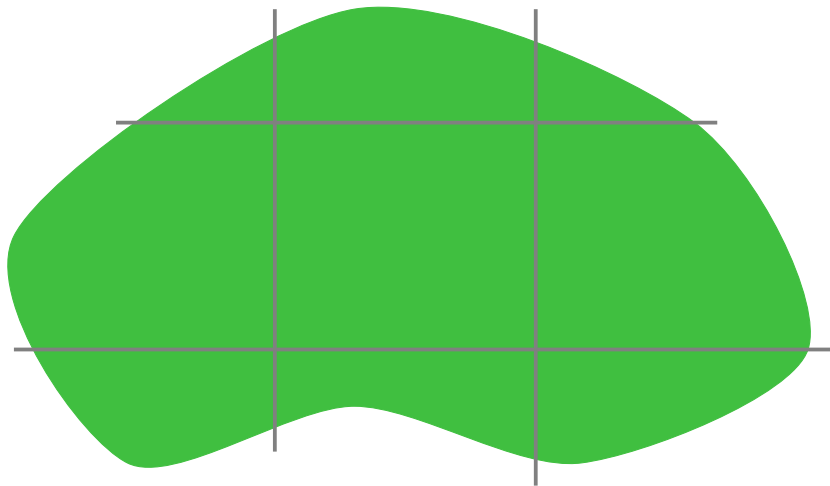
ISMP  
July 13, 2015

## Forests and stable sets



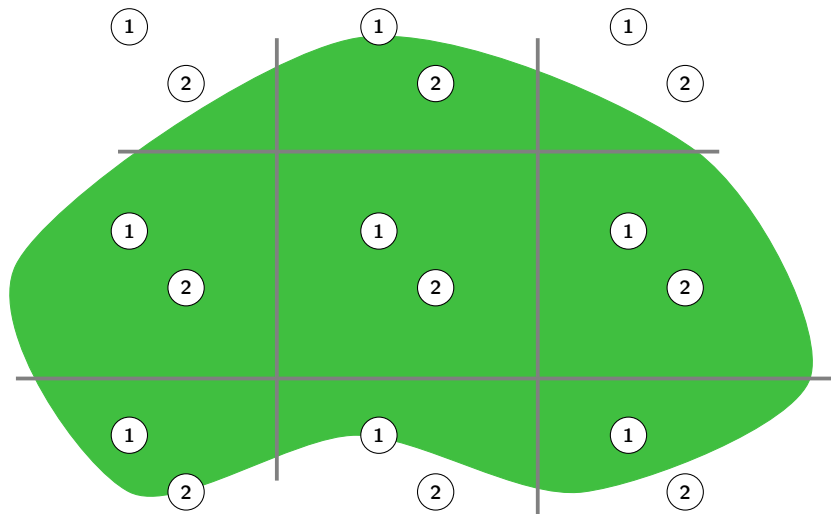
Barahona, Weintraub, and Epstein. "Habitat dispersion in forest planning and the stable set problem". *Operations Research*. 1992.

## Forests and stable sets



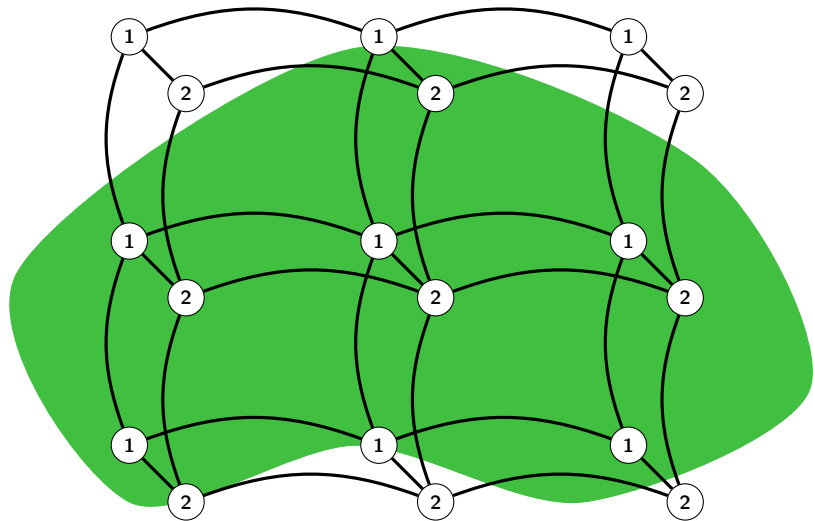
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## Forests and stable sets



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## Forests and stable sets

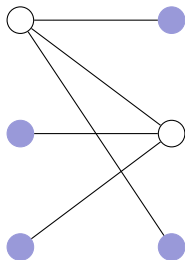


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## Interdiction overview

**Goal** Inhibit solutions to an optimization problem by limiting the feasible set

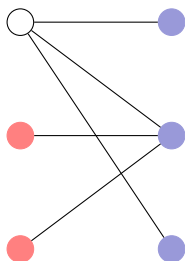
**Example** Forbid vertices to minimize the maximum stable set



## Interdiction overview

**Goal** Inhibit solutions to an optimization problem by limiting the feasible set

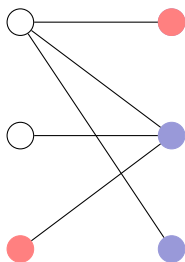
**Example** Forbid vertices to minimize the maximum stable set



## Interdiction overview

**Goal** Inhibit solutions to an optimization problem by limiting the feasible set

**Example** Forbid vertices to minimize the maximum stable set





# Our results

## This talk

Polynomial-time  $(1 + \epsilon)$ -approximation algorithm for  $b$ -stable set interdiction on bipartite graphs

## More generally

2-approximation  
or  
super-optimal with cost  $\leq 2B$

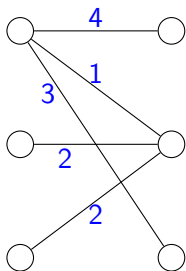
and improvements for some problems

## Extending techniques of

Burch, Carr, Krumke, Marathe, Phillips, & Sundberg. "A decomposition based pseudoapproximation algorithm for network flow inhibition." 2003.

## $b$ -stable set interdiction in bipartite graphs

$G = (V = I \cup J, E)$  bipartite graph  
 $b \in \mathbb{N}^E$  bounds

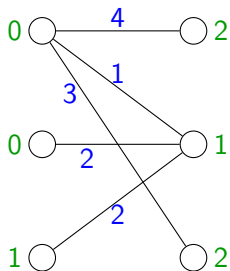


$x \in \mathbb{N}^V$  is  $b$ -stable if

$x(u) + x(v) \leq b_{uv}$ , for all  $uv \in E$ .

## $b$ -stable set interdiction in bipartite graphs

$G = (V = I \cup J, E)$  bipartite graph  
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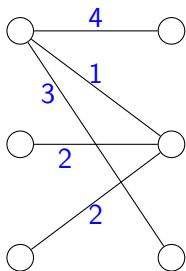


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## $b$ -stable set interdiction in bipartite graphs

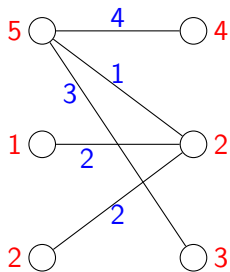
$G = (V = I \cup J, E)$  bipartite graph  
 $b \in \mathbb{N}^E$  bounds



$x \in \mathbb{N}^V$  is  $b$ -stable if

$x(u) + x(v) \leq b_{uv}$ , for all  $uv \in E$ .

# $b$ -stable set interdiction in bipartite graphs



$G = (V = I \cup J, E)$  bipartite graph  
 $b \in \mathbb{N}^E$  bounds  
 $c \in \mathbb{N}^V$  interdiction costs  
 $B \in \mathbb{N}$  interdiction budget

$x \in \mathbb{N}^V$  is  $b$ -stable if

$$x(u) + x(v) \leq b_{uv}, \text{ for all } uv \in E.$$

Interdict  $R \subseteq V$  by enforcing

$$x(v) = 0, \text{ for all } v \in R.$$

$R$  is feasible if  $c(R) \leq B$ .

# Formulation

## 1: Integer formulation

$$\begin{aligned} \text{OPT} = \min_{r \in \{0,1\}^V} \max_{c^T r \leq B} & \mathbb{1}^T x \\ & A^T x \leq b \\ & x \geq 0 \\ & r^T x = 0 \end{aligned}$$

## 2: Interdiction to the objective

$$\begin{aligned} \text{OPT} = \min_{r \in \{0,1\}^V} \max_{c^T r \leq B} & (\mathbb{1} - r)^T x \\ & A^T x \leq b \\ & x \geq 0 \end{aligned}$$

## 3: Dualize the max LP

$$\begin{aligned} \text{OPT} = \min & b^T y \\ & Ay + r \geq \mathbb{1} \\ & y \geq 0 \\ & r \in \{0,1\}^V \\ & c^T r \leq B \end{aligned}$$

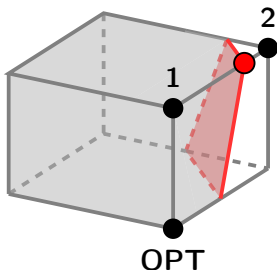
## 4: Relax

$$\begin{aligned} \text{OPT} \geq \min & b^T y \\ & Ay + r \geq \mathbb{1} \\ & y \geq 0 \\ & r \geq 0 \\ & c^T r \leq B \end{aligned}$$

## Sufficient conditions recap

- Feasible set is down-closed
- Objective is  $\max 1^T x$
- Integral LP description is box-Totally Dual Integral

# Pseudoapproximation



$$\begin{aligned} \text{OPT} \geq M = \min b^T y \\ Ay + r &\geq \mathbb{1} \\ c^T r &\leq B \\ y, r &\geq 0 \end{aligned}$$

## Pseudoapproximation guarantee

For one of  $i = 1$  or  $i = 2$

OR

$$\begin{aligned} b^T y^i &\leq 2M \quad \text{and} \quad c^T r^i \leq B \\ b^T y^i &\leq M \quad \text{and} \quad c^T r^i \leq 2B \end{aligned}$$



PTAS for bipartite  $b$ -stable set interdiction

# Edge Cover adjacency

$$\min b^T y + 0^T r + 0r_{LR}$$

$$\begin{pmatrix} A & I & 0 \\ 0 & \chi^I & 1 \\ 0 & \chi^J & 1 \end{pmatrix} \begin{pmatrix} y \\ r \\ r_{LR} \end{pmatrix} \geq \mathbb{1}$$

$$0^T y + c^T r + 0r_{LR} \leq B$$

$$y, r, r_{LR} \geq 0$$

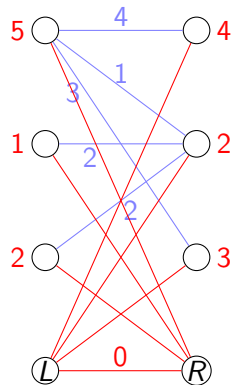
## Theorem (Hurkens)

If edge covers  $F_1, F_2 \subseteq E'$  are adj., then  $F_1 \Delta F_2$  is an alternating path or cycle.

## Augmented graph

$G' = (V \cup \{L, R\}, E')$  with

$\leftarrow$  incidence matrix



# Exploiting Edge Cover adjacency for a PTAS

## Observation 1

$$c(F_2) \leq B + 2 \max_v c(v)$$

## Observation 2

$$b(F_1) \leq \text{OPT} + 2 \max_e b(e)$$

If  $2 \max_e b(e) \leq \epsilon \text{OPT}$ , then  $F_1$  is a  $(1 + \epsilon)$ -approximation.

If not, guess the  $\lceil \frac{2}{\epsilon} \rceil$  edges in the solution with largest  $b$ -value.

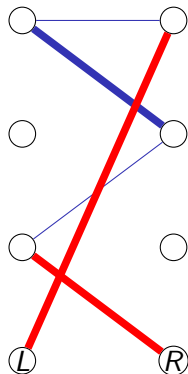
## Result

$(1 + \epsilon)$ -approximation!

## $F_1 \Delta F_2$

$F_1$ : under budget, sub-optimal

$F_2$ : over budget, super-optimal



## Interdiction summary

2-approximation or super-optimal with cost  $\leq 2B$

- Objective is maximization
- Model interdiction in the objective (down-closed)
- LP formulation with a dual integrality property &  $\{0,1\}$ -objective

Works for: matroid intersection, network flow, indep. systems with TU LP

improvements for some problems:

**$b$ -stable set**  $(1 + \epsilon)$ -approximation

**weighted rank** matroids, intersection of two matroids

**weighted rank** matroids with submodular costs

# 1-stable set interdiction in polynomial time

max stable set  $G[V \setminus R^*] = |V| - |R^*| - \text{max matching } G[V \setminus R^*]$

