Sampling Bipartite Graphs

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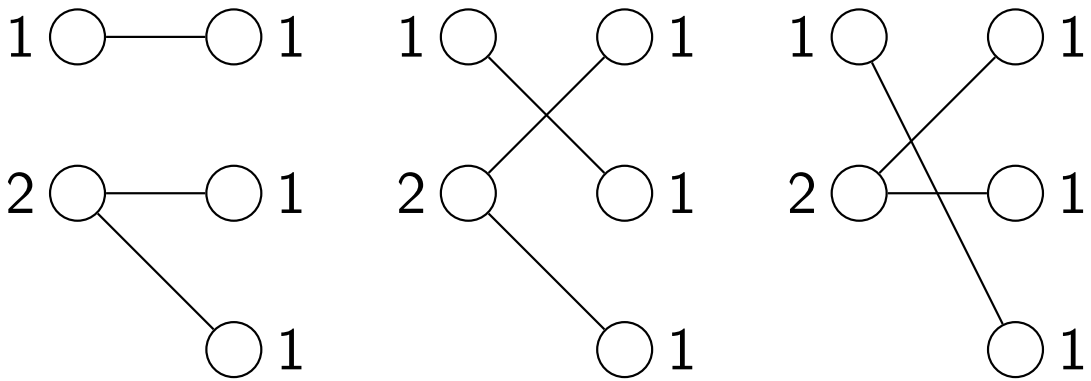
Instance

sequences $r = (r_i)_{i=1}^m$ and $c = (c_j)_{j=1}^n$

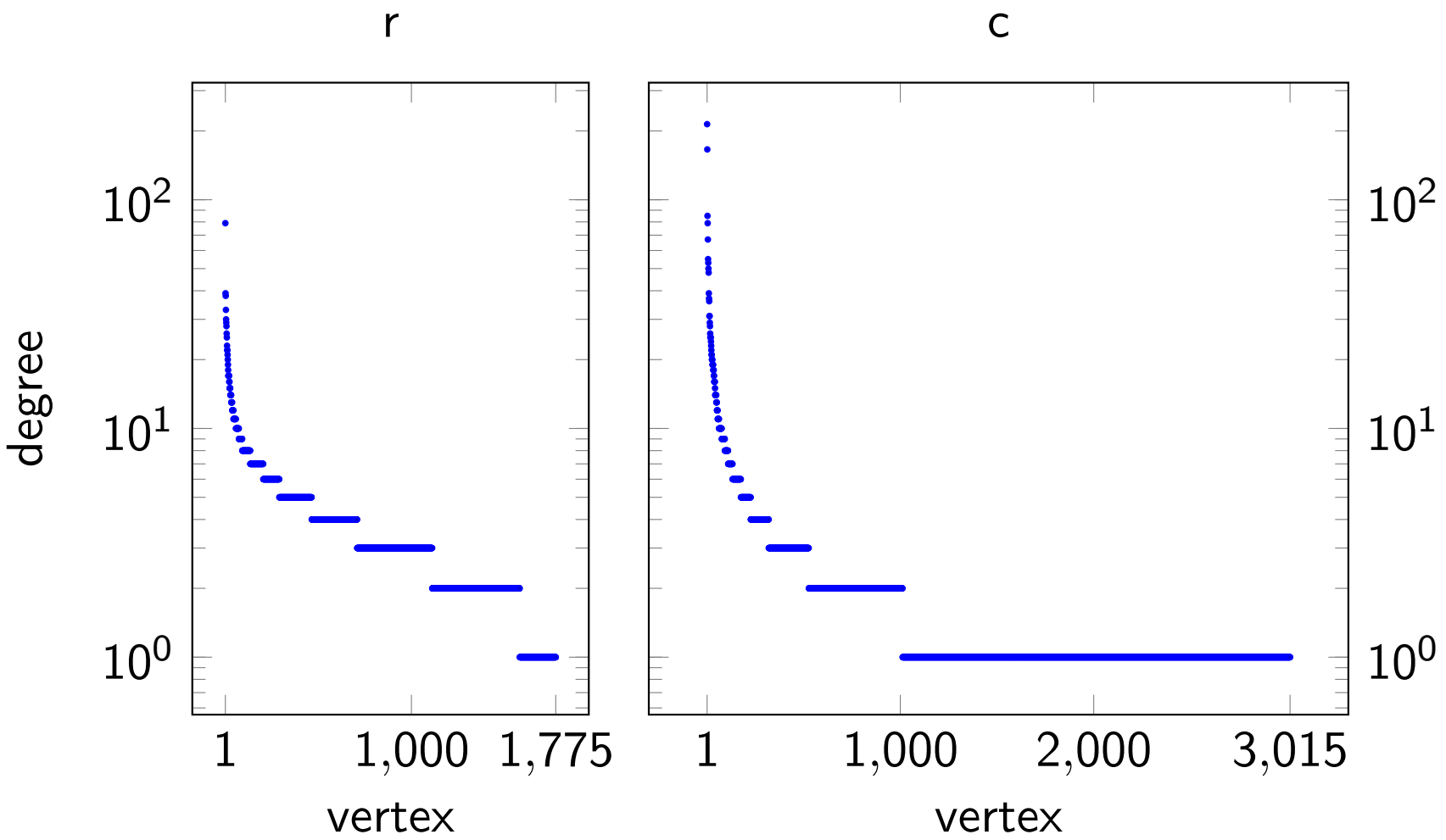
vertices $X = \{u_i\}_{i=1}^m$ and $Y = \{v_j\}_{j=1}^n$

goal Randomly sample a bipartite graph $G = (X \cup Y, E)$ that satisfies $\deg(u_i) = r_i$ and $\deg(v_j) = c_j$, for all i, j



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Typical Big Instance We Want to Solve



$n = 3015$ threads in a forum with $m = 1775$ users

Prior Work

Algorithms

Regular sequences: Kannan, Tetali & Vempala (1997)
All sequences: Bezáková, Bhatnagar & Vigoda (2007)

↓ **Practically useful** ↓

Very very sparse: Wormald (1984)
c **is bounded**: Miller & Harrison (2012)

↑ **Theoretical Guarantee** ↑

Old practitioner's standard: Chen, Diaconis, Holmes & Liu (2005)
New practitioner's standard: Miller & Harrison (2013)

Negative Results

Limits on KTV: Bezáková, Bhatnagar & Randall (2009)
CDHL may fail: Bezáková, Sinclair, Štefanovič & Vigoda (2011)

Configuration model and edge weights

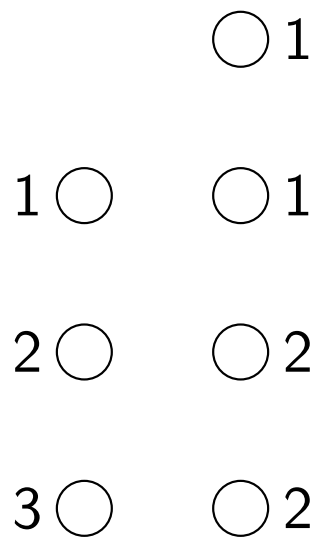
initialize Create r_i or c_i "endpoints" for each vertex

repeat Select two endpoints (one at random)

Add an edge between the vertices

Remove the endpoints

reject If the graph is not simple



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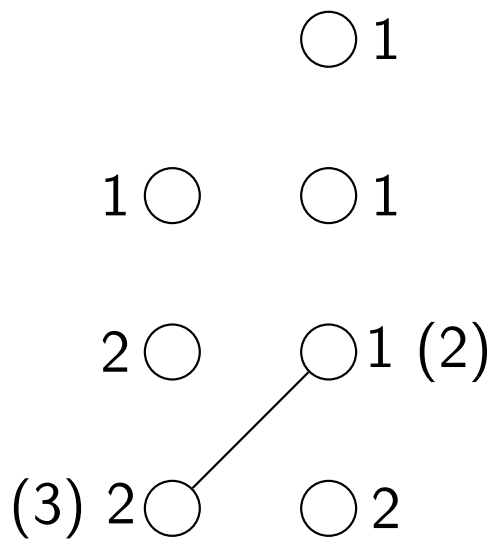

$$\frac{2}{6}$$

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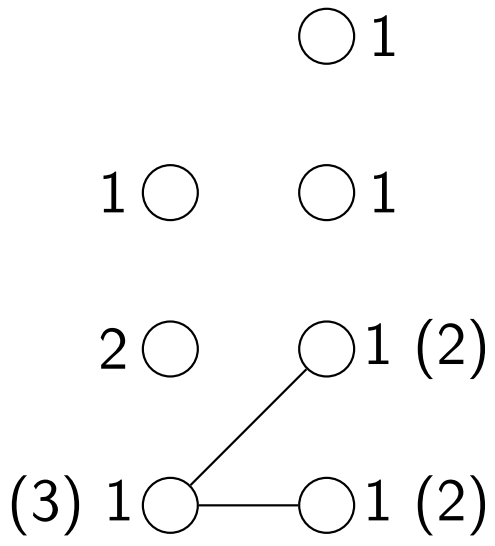

$$\frac{2}{6} \cdot \frac{2}{5}$$

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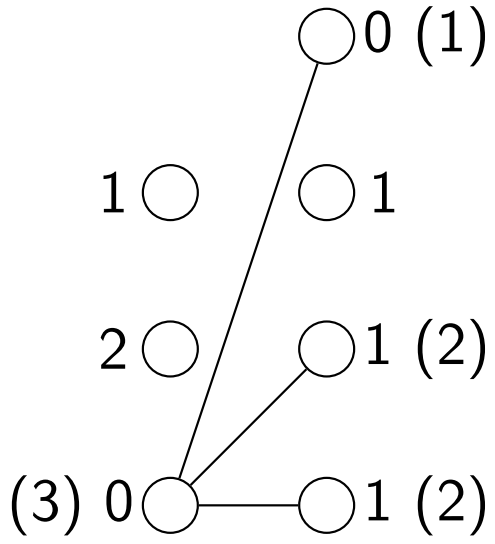

$$\frac{2}{6} \cdot \frac{2}{5} \cdot \frac{1}{4}$$

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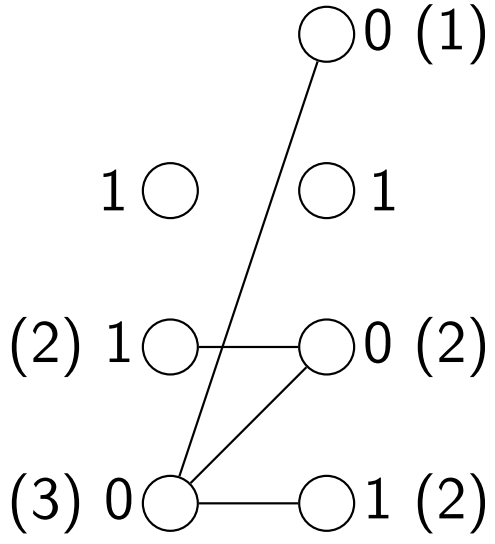

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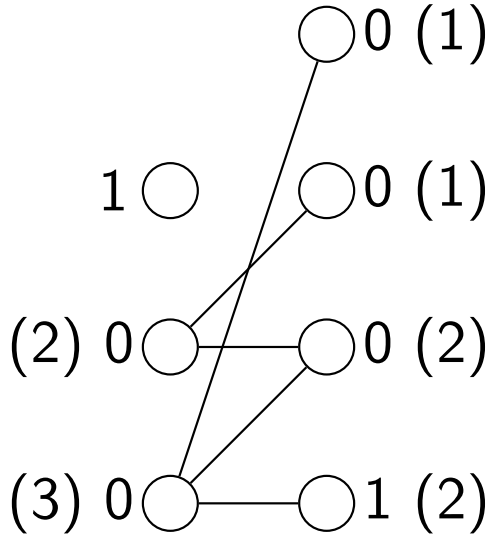

$$\frac{2}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}$$

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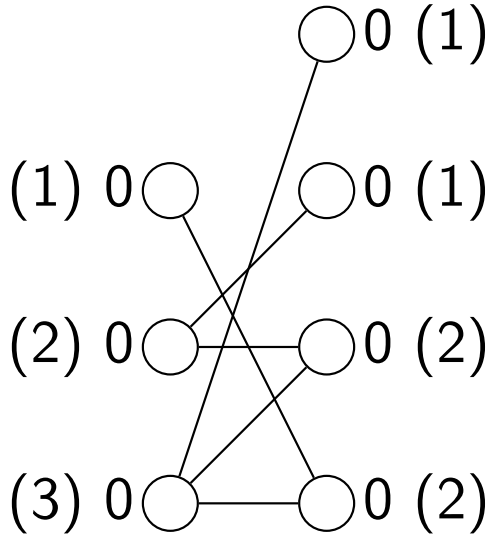
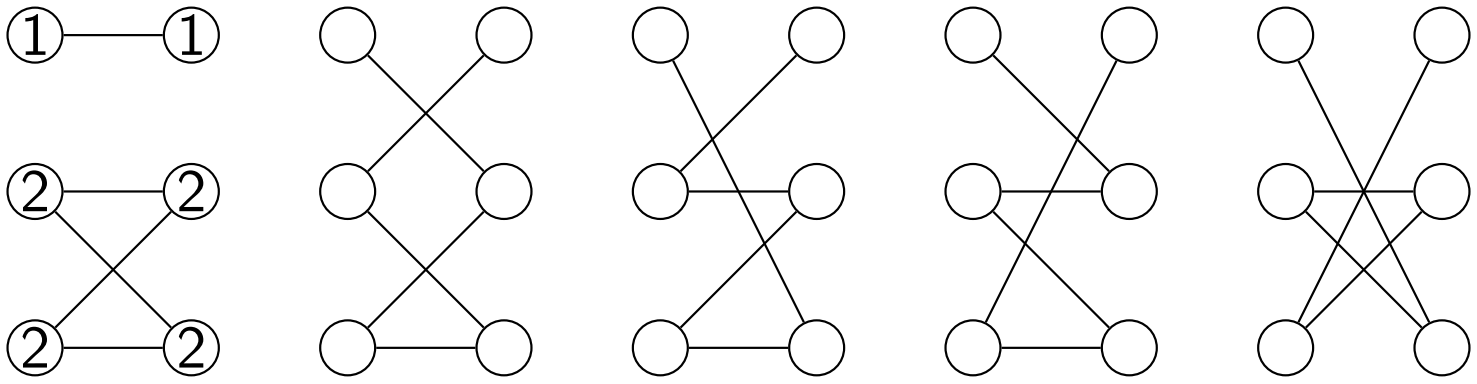


$$\frac{2}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}=\frac{2!2!1!1!}{6!}=\frac{(\prod_i r_i!)}{(\sum_i r_i)!}$$

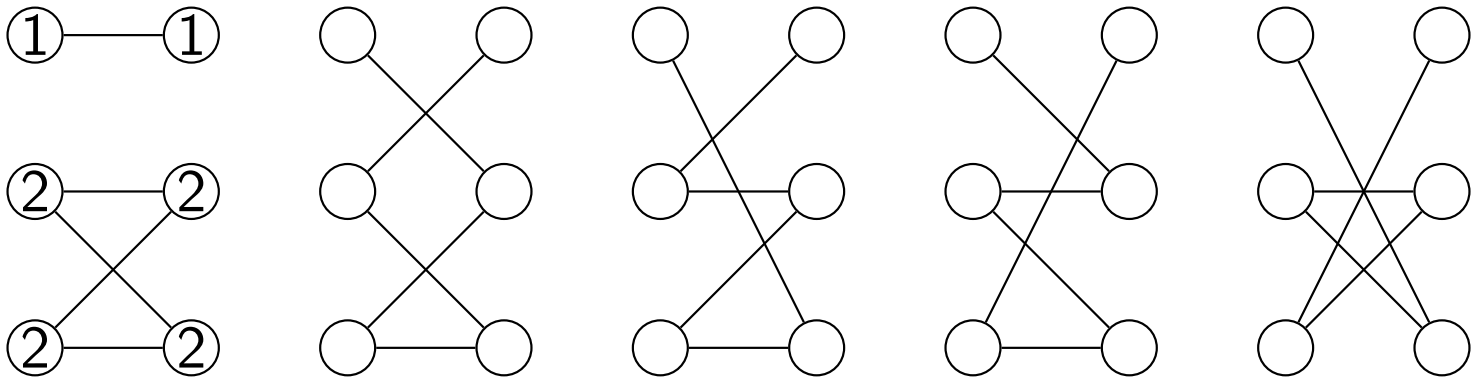
Dynamic Programming Example



1	1
2	2
2	2

- Only need to store the #vertices of each remaining #endpoints
- Initialization complexity is $O\left(n^{2\max_j c_j}\right)$

Dynamic Programming Example



$$\begin{matrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{matrix} \times 1 \rightarrow \begin{matrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{matrix}$$

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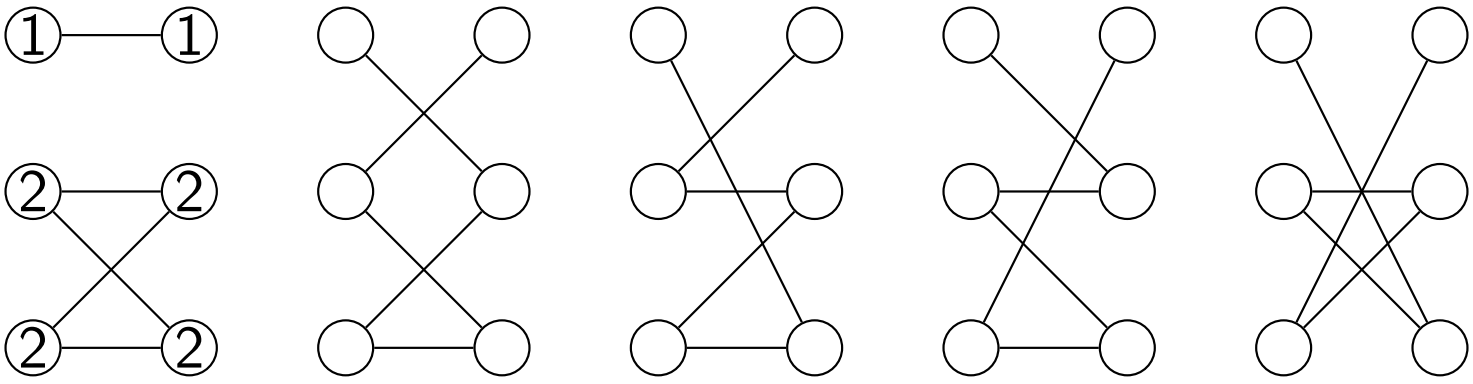
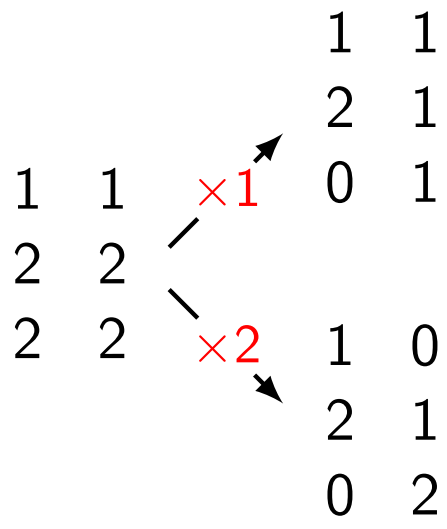
Dynamic Programming Example





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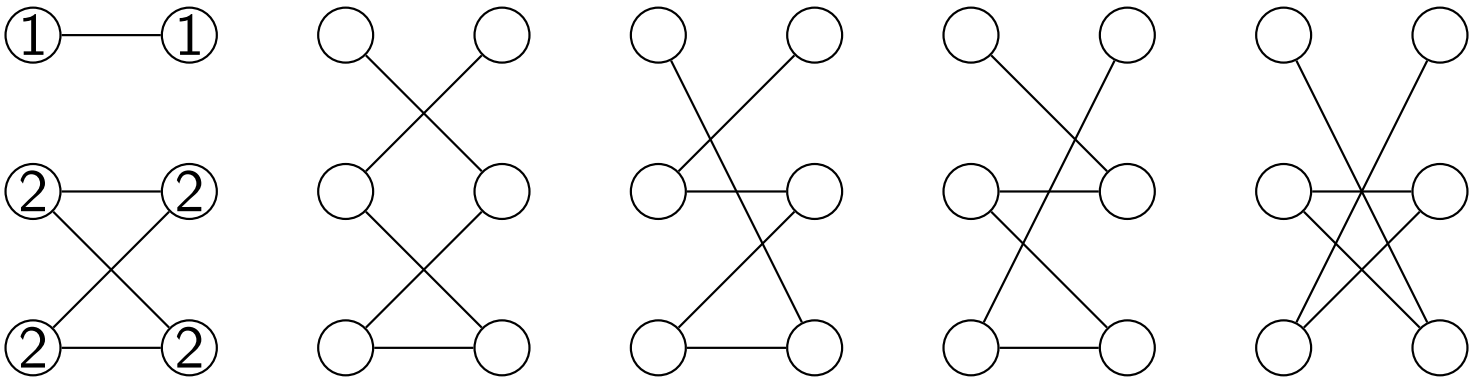

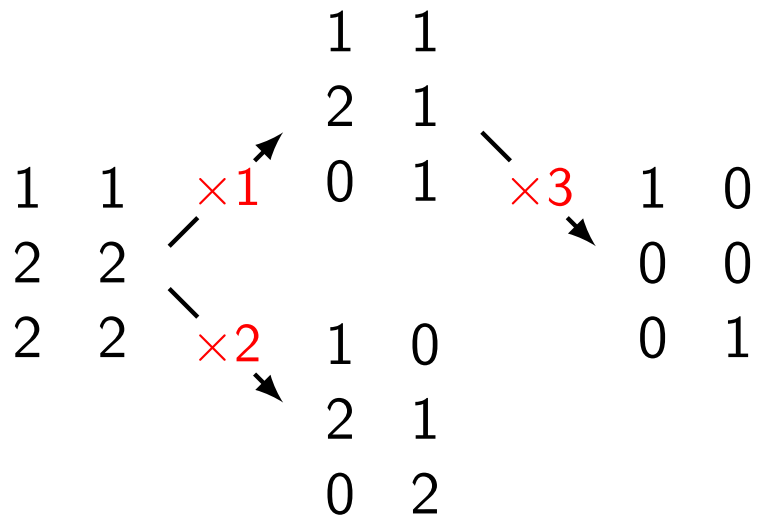

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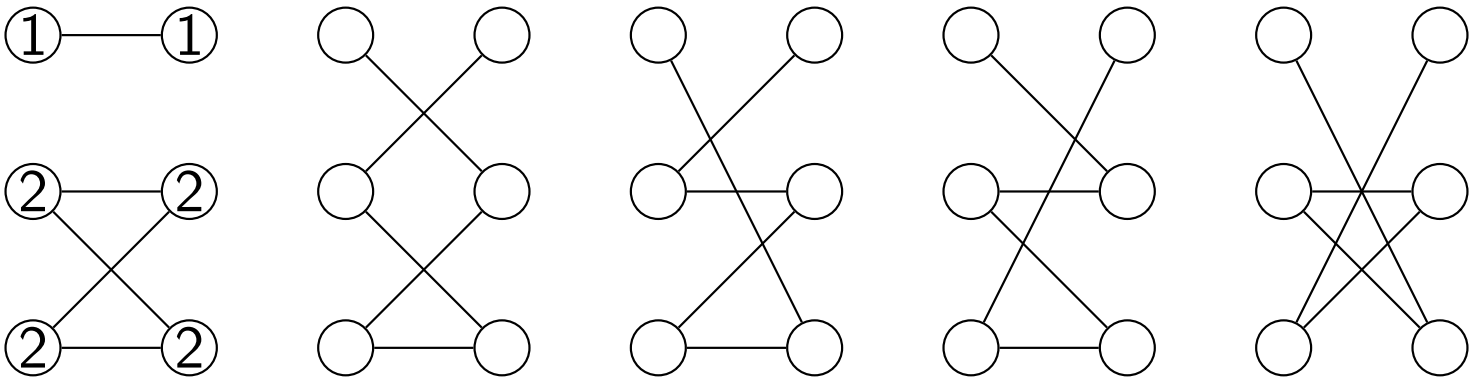

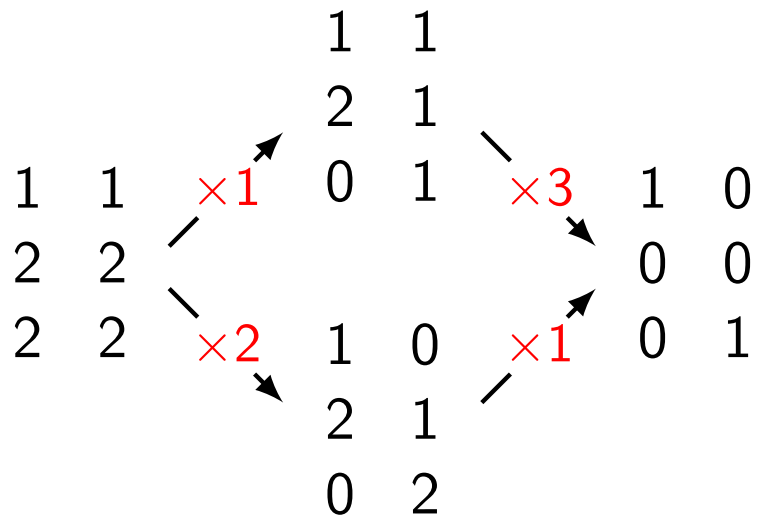

- Only need to store the #vertices of each remaining #endpoints
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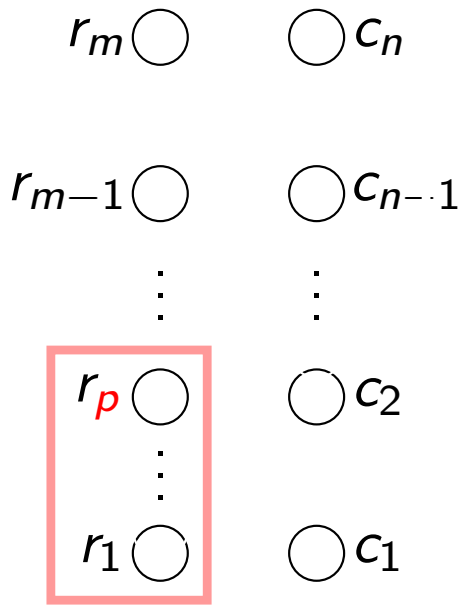
Dynamic Programming + Configuration Model

Hybrid Algorithm

choose Integer $p \geq 1$

1. Dynamic programming of the p special vertices
2. Configuration model for remainder



Effect

Samples from configuration model **conditionally given** no repeated edges to p special vertices.

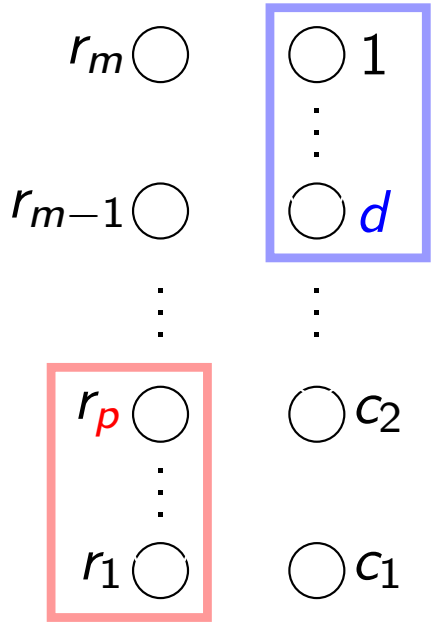
Approximate Dynamic Programming + Config. Model

Hybrid Approximation Algorithm

choose Integers $p \geq 1$ and $d \geq 0$

approx If $\deg(v_j) > d$ approx. #endpoints of v_j by c_j

2. Dynamic programming for p special vertices
3. Rejection step
4. Configuration model for remainder



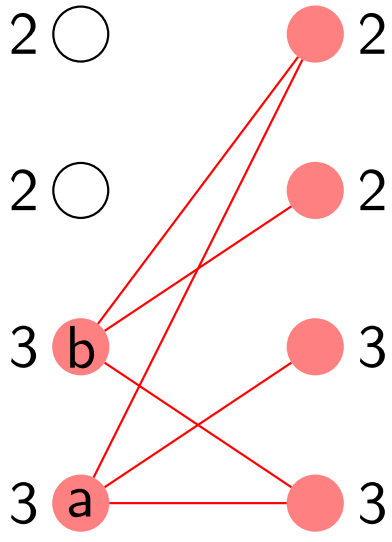
Analysis

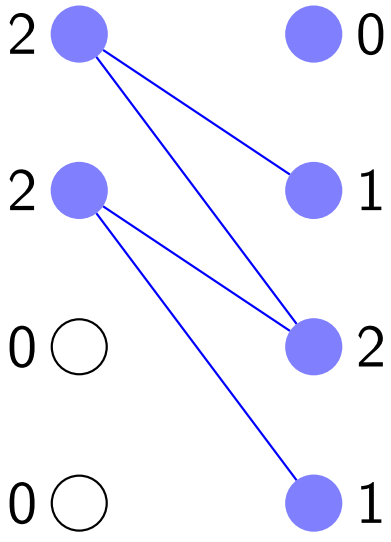
complexity Initialization complexity is $O(n^{2(d+1)})$

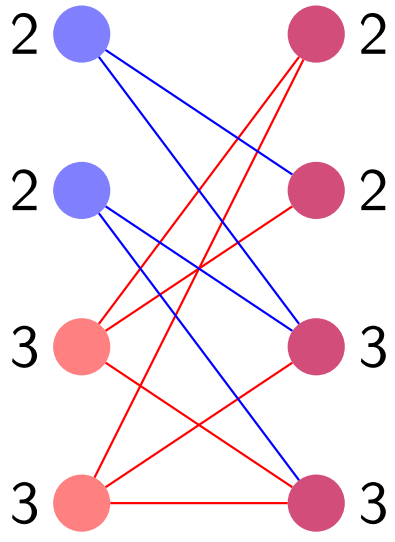
output Sample distribution is exactly uniform

Example $p = d = 2$

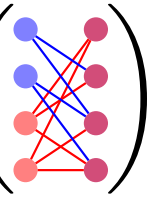
$$\frac{\overbrace{(3 \cdot 3 \cdot 2)}^{\text{vertex a}}\overbrace{(3 \cdot 2 \cdot 1)}^{\text{vertex b}}}{1980} \underbrace{\binom{2}{3}}_{\text{accept}} \overbrace{\left(\frac{2!^3}{4!}\right)}^{\substack{\text{config.} \\ \text{model}}} = \frac{3!^2 2!^4}{1980 \cdot 4!}$$







Example continued

	$P\left(\text{  }\right)$			$P(\text{Reject})$
Configuration Model	$\frac{3!^4 2!^4}{10!}$	$\approx$	0.0057	0.81
Hybrid algorithm	$\frac{3!^2 2!^4}{1980 \cdot 4!}$	$\approx$	0.012	0.59
Uniform	$\frac{1}{34}$	$\approx$	0.029	0

Conclusions

- Hybrid algorithm extends the Configuration Model and Dynamic Programming algorithms.
- Doesn't work (yet) for the big instance shown earlier
- Best chance is with sequences with many low degrees, a few high degrees